Calibration and Simulation of DSGE Models*

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cal·i·bra·tion (kăl′ə-brā′shən) *n.* The process of restricting parameters in an economic model so that the model is consistent with long run growth facts and microeconomic observations.

1 Introduction

Many interesting macroeconomic models are either sufficiently complex that they must be solved computationally, or the questions being asked are inherently quantitative and so they should be solved computationally. The first group includes almost any empirically relevant version of the neoclassical growth model.[1] The second group includes such basic questions as business cycle fluctuations: How well does the the neoclassical growth model do in producing variation in macroaggregates (like output, consumption, investment and hours worked) that "look like" those seen in the data. These are quantitative questions for which qualitative answers are insufficient. Calibration is an effective tool for imposing discipline on the choice of parameter values that arise in such models, taking what would otherwise be a numerical example into the realm of an empirically relevant exercise with parameters tightly pinned

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[1]Except in very special cases, the neoclassical growth model does not allow for closed-form solutions (that is, ones that can be worked out by hand). Perhaps the best known of these is when utility is logarithmic in consumption, separable between consumption and hours worked, the production function is Cobb-Douglas, and depreciation is 100%. These restrictions are very special, and in the case of depreciation, clearly at odds with the data.

down by either long run growth facts, or microeconomic observations. As such, calibration is a useful part of the macroeconomist's toolkit.

This chapter is concerned with measurement as it pertains to calibration.[2] Kydland and Prescott (1982) provided the foundations for the calibration procedure; key subsequent developments have been made by Prescott (1986), Cooley and Prescott (1995) and Gomme and Rupert (2007). This chapter builds chiefly on Gomme and Rupert. Like this earlier paper, our goal is to provide a sufficiently careful and detailed description of our procedures that others will be easily able to replicate our work. To further facilitate replication, the underlying data and manipulations are available at `http://alcor.concordia.ca~pgomme`.

In order to make the presentation as "hands-on" as possible, Section 2 presents the model, the neoclassical growth model, which forms the foundation of New Classical and New Keynesian models. The heart of the paper is in Section 3 which presents the calibration of the neoclassical growth model. The first order of business is to choose functional forms for preferences and technology. These choices are guided, in part, by long run growth considerations. A second key issue is how broadly economic activity should be measured. Cooley and Prescott (1995) think of economic activity very broadly, including not only private market activity but also government and household activity. Gomme and Rupert (2007) focus more narrowly on private market activity. However, as argued in Section 3.2, for Cooley and Prescott to aggregate economic activity across the private market, government, and household sectors, it must almost certainly be true that each of these sectors has the *same* technology. This chapter follows Gomme and Rupert, focusing narrowly on private market activity since it involves fewer imputations.[3] It is important to remember that if the Cooley and Prescott aggregation is correct, then the Gomme and Rupert measurement should give the same answer as Cooley and Prescott; the converse is not true. Much of the remainder of Section 3 describes the nitty gritty details of constructing various calibration targets. The chapter describes how to actually construct more calibration targets than are necessary to calibrate the neoclassical growth model. Section 3.12 describes how to actually calibrate the model which involves making sure that in steady state, the model is consistent with the calibration targets. It is important to remember that calibration is a process for mapping a set of calibration targets into an identical number of model parameters; it is not simply

[2]Measurement is also important in establishing the basic facts to be explained. A basic set of data and business cycle moments are included with the accompanying Matlab/Octave program file, available at `http://alcor.concordia.ca~pgomme`.

[3]As discussed in Section 3.2, Cooley and Prescott (1995) must impute government capital income since it is missing from National Income and Product Accounts (NIPA). They should also impute labor income to owner occupied housing (the capital income is already in NIPA), but do not. Given their broad interpretation of economic activity, in calibrating their model, Cooley and Prescott should include time spent on housework, but do not. As discussed in Section 3.7, individuals spend a considerable amount of time on housework.

setting parameter values.

The model is solved and simulated in Section 4. Rather than the usual real business cycle practice of comparing two tables of second moments – one for the U.S. data, the other for model-generated data – Solow residuals as measured from the U.S. data are fed into the model as the set of technology shocks. The model is, then, evaluated on its ability to generate time series that are similar to those in the data. The model does reasonably well in replicating the time series behavior of output, consumption and average labor productivity; it does rather poorly with respect to hours, investment and capital.

2 The Economic Environment

The organizing framework is the neoclassical growth model that lies at the heart of New Classical and New Keynesian models. The presentation is kept relatively brief since this model should be familiar to most macroeconomists. It should be understood that the variables chosen by households differ from those chosen by firms, and that these differ from aggregate or per capita quantities; in the presentation, no distinction is made between these three sets of variables in the interests of conserving on notation.

Also in the interests of a clean presentation of the model, growth is omitted. There are two logical ways for growth to appear: as labor-embodied technological progress, and in the form of investment-specific technological change as in Greenwood et al. (1997).[4] As shown in King et al. (1988) and Gomme and Rupert (2007), including growth is important for delivering certain parameter restrictions in both preferences and technologies. These parameter restrictions are discussed below.

2.1 Households

The problem of the representative household is

$$\max E_0 \sum_{t=0}^{\infty} \beta^t U(c_t, \ell_t), \quad 0 < \beta < 1 \tag{1}$$

subject to a budget constraint,

$$c_t + x_t = (1 - \tau_n) w_t n_t + (1 - \tau_k) r_t k_t + \tau_t, \tag{2}$$

[4] Except in the special case of a Cobb-Douglas production function, disembodied technological change is not consistent with balanced growth.

the law of motion for capital,

$$k_{t+1} = (1-\delta)k_t + x_t, \quad 0 \le \delta \le 1, \tag{3}$$

and a constraint on time,

$$n_t + \ell_t = 1. \tag{4}$$

The household's preferences are defined over contingent time sequences for consumption, c_t, and leisure, ℓ_t. In the budget constraint, Eq. (2), the real wage is w_t while the rental rate for capital is r_t. Labor income is taxed at the rate τ_n while capital income is taxed at the rate τ_k.[5] Hours of work are denoted n_t while k_t is the household's beginning-of-period holdings of capital. The household receives a lump-sum transfer of τ_t from the government. Finally, x_t is the household's investment and δ is the depreciation rate of capital.

2.2 Firms

Firms face a sequence of static problems. Each period, a firm hires labor and rents capital to maximize its real profits:

$$\max_{k_t, n_t} F(k_t, n_t; z_t) - w_t n_t - r_t k_t \tag{5}$$

where F is a constant-returns-to-scale production function, and z_t is a shock to technology.

In the New Keynesian literature, it is common to assume two sectors, one for final goods, the other for intermediate goods. The final goods sector is perfectly competitive and uses only intermediate goods. The intermediate goods sector is characterized by monopolistic competition, and production employs labor and maybe capital. The reason for the two sector setup is because most New Keynesian models include sticky price setting which means that at least some firms must be price setters. The complications of the New Keynesian setup are suppressed in the interests of clarity.

2.3 Government

The only role for government is to levy distorting factor income taxes, lump-sum rebating the proceeds. In particular, there is no government spending and the government issues no debt. Its budget constraint is

$$\tau_t = \tau_n w_t n_t + \tau_k r_t k_t. \tag{6}$$

[5]The tax on capital is measured net of the capital consumption allowance.

3 Calibration

At this stage, the task is to choose functional forms, then find calibration targets that can be used to assign values to the parameters of the model.

3.1 Functional Forms

To be consistent with balanced growth, the momentary utility function, U, needs to be homogeneous of some degree in consumption, c. For the most part, the real business cycle literature uses

$$U(c,\ell) = \begin{cases} \frac{(c\ell^{\omega})^{1-\gamma}}{1-\gamma} & \text{if } \gamma \in (0,1) \cup (1,\infty), \\ \ln c + \omega \ln \ell & \text{if } \gamma = 1. \end{cases} \tag{7}$$

This utility specification is referred to as "constant relative risk aversion." Above, U is homogeneous of degree $(1-\gamma)$ in c; in other words, this utility function satisfies balanced growth restrictions. In the New Keynesian literature, it is more common to see

$$U(c,n) = \ln c - \omega \frac{n^{1+\xi}}{1+\xi} \tag{8}$$

where $1/\xi$ is the Frisch labor supply elasticity.

The production function is specified to be Cobb-Douglas:

$$F(k,n;z) = zk^{\alpha}n^{1-\alpha}, \quad 0 \leq \alpha \leq 1. \tag{9}$$

In the literature, the Cobb-Douglas functional form is often justified as being consistent with the following facts:

1. capital's share of income exhibits no secular trend,
2. the return to capital similarly has no secular trend,
3. the real wage rate does have a secular trend.

Swan (1964), Phelps (1966) and King et al. (1988) show that *any* constant-returns-to-scale production function is consistent with these facts. What makes the case for Cobb-Douglas more compelling is to incorporate investment-specific technological change; Gomme and Rupert (2007) make explicit arguments made in Kydland (1995) and Greenwood et al. (1997), and the interested reader is directed to these works for details.

Finally, the productivity shock is assumed to follow a first-order autoregressive process:

$$\ln z_t = \rho \ln z_{t-1} + \epsilon_t, \quad \epsilon_t \sim N(0,\sigma^2). \tag{10}$$

Table 1: Parameters to be calibrated

Preferences:	β, ω, γ or ξ
Technology:	δ, α, ρ, σ
Government:	τ_n, τ_k

The set of parameters to be calibrated (assigned values) is summarized in Table 1. Since there are 9 parameters, procedurally calibration involves the use of 9 calibration targets. These targets are typically taken from two sources: microeconomic evidence, and long run growth facts. Long run growth facts have already implicitly been used to restrict the production function to be Cobb-Douglas, as well as to restrict the utility function. There is an embarrassment of riches since there are more potential calibration targets than parameters.[6] The properties of the technology shock, ρ and σ, and be inferred from the properties of the Solow residual; see Section 3.10. The tax rates, τ_n and τ_k, can be obtained directly from National Income and Product Accounts (NIPA) as described in Section 3.8. The remaining 5 parameters can be calibrated using some subset of:

1. Factor income shares using data from NIPA.

2. Depreciation and capital stock data reported by the Bureau of Economic Analysis (BEA).

3. The investment-output ratio.

4. The capital-output ratio.

5. Microeconomic evidence concerning risk aversion.

6. Microeconomic evidence regarding the labor supply elasticity.

7. The allocation of time from U.S. time use surveys.

8. The real return to capital.

The remainder of this section discusses the measurement of the above calibration targets.

3.2 How Broadly to Measure Economic Activity

An issue that immediately arises is how broadly one should measure economic activity for the purposes of calibration. Perhaps the best known paper on calibration is that of Cooley

[6]The surplus of calibration targets could be handled using formal econometric techniques that take advantage of the "over identification" of the parameters.

and Prescott (1995). They construe economic activity very broadly, encompassing both private market activity, government production, and home activity. Cooley and Prescott include capital in all these sectors, plus the stock of inventories and the value of land, in their measure of the capital stock; output likewise includes output produced by this capital stock. In measuring income flows, Cooley and Prescott must impute capital income flows to government capital since NIPA does not include a measure of government capital income. For similar reasons, they must impute income flows to the stock of consumer durables. Oddly, they do not impute labor income flows for the housing sector, although these flows are largely missing from NIPA. Likewise, Cooley and Prescott's measure of time spent working only includes market time; see the discussion in Section 3.7

An alternative approach is that exemplified by Gomme and Rupert (2007) who measure economic activity more narrowly, focusing on private market activity, when it comes to calibration. If Cooley and Prescott are justified in their aggregation, it should not matter whether one takes the broad or narrow approach to measuring economic activity. To see this point, let sectoral outputs are given by

$$Y_M = K_M^{\alpha} N_M^{1-\alpha} \tag{11}$$

$$Y_G = K_G^{\alpha} N_G^{1-\alpha} \tag{12}$$

$$Y_H = K_H^{\alpha} N_H^{1-\alpha}, \tag{13}$$

where an M subscript denotes private market activity, G government activity, and H the household sector. For the Cooley and Prescott aggregation to be valid, it must be the case that all three production functions have the same capital share parameter, α; otherwise, they surely would not be able to write the aggregate production function as

$$Y = \left(K_M + K_G + K_H\right)^{\alpha} \left(N_M + N_G + N_H\right)^{1-\alpha}. \tag{14}$$

To find a value for capital's share, α, it should not matter whether one uses Eq. (11), as Gomme and Rupert do, or Eq. (14), as Cooley and Prescott do. Since Cooley and Prescott must impute various income flows while Gomme and Rupert do not, it seems more straightforward to follow the narrow approach of Gomme and Rupert.

Further, there are good reasons for thinking that the three sectors are *not* sufficiently similar to aggregate as in Eq. (14). For example, the home production literature emphasizes that there is an inherent asymmetry between the market and home sectors: the market sector produces goods that are used in the home sector (namely, investment goods including durables), but the home sector does not produce goods that are used by the market sector; see Benhabib et al. (1991) and Greenwood and Hercowitz (1991).

3.3 Capital's Share of Income

In principle, one of the easiest parameters to calibrate is α, capital's share of income. Given that the production function is Cobb-Douglas, and assuming that factor markets are competitive, factors are paid their marginal products. Capital's share of income can, then, be computed as total payments to capital divided by income. In practice, the calculation is far from straightforward, as already suggested by the discussion of the different approaches of Cooley and Prescott (1995) and Gomme and Rupert (2007).

A further issue relates to the treatment of proprietors' income and indirect taxes less subsidies. The problem is that both go into measured GDP, and both have capital and labor income components that cannot easily be separated out. To see how to proceed, write out market income as:

$$Y_M = Y_{KM} + Y_{NM} + Y_{AM} \tag{15}$$

where Y_{AM} denotes ambiguous market income, namely proprietors' income plus indirect taxes less subsidies, Y_{KM} is unambiguous capital income, and Y_{NM} is unambiguous labor income. Some portion of ambiguous income needs to be allocated to capital income, the rest to labor income. The practice in the literature is to assume that the fraction of ambiguous income that should be allocated to capital is the same fraction as income is allocated to capital for the rest of the economy. This idea is, perhaps, clearer when stated as an equation,

$$Y_{KM} + \alpha Y_{AM} = \alpha Y_M, \tag{16}$$

where α is the (unknown) capital share. The left-hand side is *total* capital income, including a fraction of ambiguous income; the right-hand side is capital income as a share of total income. Eq. (16) can be rewritten as

$$\alpha = \frac{Y_{KM}}{Y_{KM} + Y_{NM}}. \tag{17}$$

As discussed above, one complication is that market income flows are 'contaminated' by housing income flows. Fortunately, NIPA includes data on the housing sector. Market capital income is given by

$$
\begin{aligned}
Y_{KM} = & \text{Rental Income} - \text{Housing Rental Income} \\
& + \text{Net Interest Income} - \text{Housing Net Interest Income} \\
& + \text{Corporate Profits} - \text{Housing Corporate Profits} \\
& + \underbrace{\text{Gross National Product} - \text{Net National Product}}_{\text{Total Consumption of Fixed Capital}} \\
& - \underbrace{(\text{Government Gross Value Added} - \text{Government Net Domestic Product})}_{\text{Government Consumption of Fixed Capital}} \\
& - \underbrace{(\text{Housing Gross Value Added} - \text{Net Housing Value Added})}_{\text{Housing Consumption of Fixed Capital}}.
\end{aligned} \tag{18}
$$

In Eq. (18), consumption of fixed capital (with appropriate adjustments to remove flows associated with the government and housing sectors) reflects compensation to capital for depreciation.

Market labor income is:

$$
\begin{aligned}
Y_{NM} = & \text{Compensation of employees} - \text{Housing compensation of employees} \\
& - \text{Government compensation of employees.}
\end{aligned} \tag{19}
$$

Capital's share of income is, then, computed via Eq. (17).

3.4 The Depreciation Rate

The Bureau of Economic Analysis (BEA) reports total depreciation for various categories of capital goods, as well as capital stocks. The depreciation rate can, then, be obtained by dividing (nominal) depreciation by the (nominal) capital stock. The depreciation rate on market capital uses data on private equipment & software and private nonresidential structures.

For those interested in modeling the home sector, the corresponding categories for computing the depreciation rate for home capital are: private residential fixed assets (structures), and the stock of consumer durables.

Depreciation rates are reported below for market and home capital, as well as their chief components.

3.5 Great Ratios

Two of the so-called "great ratios" are the investment-output and capital-output ratios. Given the discussion of the measurement of capital's share of income, α, it makes sense that output should correspond to private output (that is, excluding government), net of housing. Investment, then, should include investment in private nonresidential structures, and in equipment & software. Inventory investment is excluded from total investment because very few macroeconomic models explicitly model inventories. Output corresponds to the sum of private investment and consumption of nondurables and services.

Measuring the capital-output ratio is fraught with not only similar issues to those for investment, but others unique to the measurement of capital. Specifically, what exactly comprises private market capital? It seems clear that the stock of nonresidential structures, and equipment & software should be included. It should be noted, however, that the BEA's inclusion of software in 'capital' is a relatively recent decision. One can make a case for inventories on the basis that NIPA includes changes in the stock of inventories as part of investment. A case can also be made for including land. The problem with land is that its value is computed as a residual from the flow of funds accounts by the Board of Governors of the Federal Reserve System, and the value of land for the U.S. as a whole is sometimes found to be *negative*. In any event, the Board of Governors no longer reports the value of land.

Putting aside these issues, a further problem with capital is that its measurement has been subject to somewhat infrequent but large revisions as reported in Herman (2000). For broadly defined measures of capital, in 1997 the BEA revised their estimates of the capital stock up by as much as 30%. Alternative measures of the U.S. capital stock, like those of Maddison (1995), give estimates that are even larger. In light of these issues regarding the measurement of capital, using the capital-output ratio as a calibration target seems unwise.

3.6 Microeconomic Evidence

Microeconomic evidence can be brought to bear on two calibration targets: the coefficient of relative risk aversion, and the labor supply elasticity. Since Hall (1978), empirical work on consumption has used an intertemporal Euler equation to estimate key parameters of the utility function. Using this approach, Dynan (1993) reports estimates of the elasticity of intertemporal substitution near 0.1, or a coefficient of relative risk aversion of 10. In contrast, Gruber (2006), using Consumer Expenditure Survey data on total non-durable consumption, estimates a value of around 2 which implies a coefficient of relative risk aversion of 0.5. Using aggregate data, Attanasio and Weber (1995) find an elasticity of intertemporal substitution of either 0.34 or 0.48 – a coefficient of relative risk aversion of roughly 3 or 2. These finds are generally in line with evidence surveyed by Mehra and Prescott (1985); they conclude that the coefficient of relative risk aversion is positive, and restrict its value to be no larger than 10, although the bulk of the evidence points to a smaller value. In the literature, it is fairly common to implicitly set $\gamma = 1$ by assuming logarithmic utility.

Micro evidence on the labor supply elasticity typically use data on men. Typically, this labor supply elasticity is found to be small but positive. Altonji (1986) estimates an elasticity no larger than 0.35; MaCurdy (1981) no larger than 0.5. Pencavel (1986) surveys the empirical literature; he finds that estimates of the male labor supply elasticity are generally less than 1/3. For the logarithmic case, Eq. (7) implies a (steady state) Frisch labor supply elasticity given by $\frac{1-h}{h}$. For the largest estimated labor supply elasticity, 0.5, the Frisch labor supply elasticity means that steady state hours are 2/3 of the time endowment; smaller labor supply elasticities correspond to larger fractions of the time endowment. As shown in Section 3.7, these fractions are not consistent with U.S. time use evidence. When the utility function is given by Eq. (8), it seems common to assume a Frisch labor supply elasticity of 1 which would imply $\xi = 1$. There is nothing intrinsic to Eq. (8) to preclude setting the labor supply elasticity parameter equal to that estimated by labor economists, then use the parameter ω to ensure that hours worked are consistent with the time use survey evidence.

3.7 Time Use Surveys

One of the most comprehensive measurements of time use in the U.S. are the periodic time use surveys which were taken in 1965, 1975, 1985, 1995 and most recently the American Time Use Survey (ATUS), starting in 2003. Based on the ATUS, Gomme and Rupert (2007) report that individuals aged 16 and older spend 25.5% of their discretionary time (that is, excluding time for sleeping and other personal care) working in the market. This fraction is considerably smaller than the value of 1/3 typically used in the literature. Gomme and Rupert compute a higher fraction, 31.5%, for individuals aged $16 - 64$. However, since the majority of macroeconomic models are of a representative agent, there seems to be no good reason to exclude retirees from the calculation of the fraction of time spent working.

Curiously, Cooley and Prescott (1995) calibrate to a working time fraction of 1/3. Given their broad notion of economic activity as including the home sector, it would make sense for them to include both time spent working in the market and time spent working at home. For the 16+ population, Gomme and Rupert (2007) find that an average of 24% of discretionary time is spent performing housework; for those aged $16 - 64$, 25.1%. Arguably, then, Cooley and Prescott should have calibrated to an average work time of 49.5% (the 16+ population) or 56.6% (the $16-64$ population).

3.8 Taxes

The calculation of tax rates follows the methodology of Mendoza et al. (1994) and Carey and Tchilinguirian (2000); see also Gomme et al. (2011). Auray et al. (2011) construct tax rates for the U.S. and a subset of the E.U. The first step is to compute the tax rate on general household income, denoted τ_h (as distinct from the tax on earnings, τ_n), as the ratio of total household taxes divided by total household income:

$$\tau_h = \frac{\text{Personal Current Taxes}}{\text{Net Interest} + \text{Proprietors' Income} + \text{Rental Income} + \text{Wages and Salaries}} \tag{20}$$

Next, the tax rate on earnings is obtained as

$$\tau_n = \frac{\text{Labor Income Taxes}}{\text{Labor Income}} \tag{21}$$

where

$$\begin{aligned} \text{Labor Income Taxes} =& \tau_h \left[\text{Wages and Salaries} + (1-\alpha)\text{Proprietors' Income}\right] \\ &+ \text{Contributions for Government Social Security}, \end{aligned} \tag{22}$$

and

$$\begin{aligned} \text{Labor Income} =& \text{Wages and Salaries} + (1-\alpha)\text{Proprietors' Income}] \\ &+ \text{Employer Contributions for Government Social Security}. \end{aligned} \tag{23}$$

In the above, α is capital's share of income; see Section 3.3.

Finally, the capital income tax rate is given by

$$\tau_k = \frac{\text{Capital Income Taxes}}{\text{Capital Income}} \tag{24}$$

where

$$\begin{aligned} \text{Capital Income Taxes} =& \tau_h \left[\text{Net Interest} + \alpha \text{ Proprietors' Income} + \text{Rental Income}\right] \\ &+ \text{Corporate Income Taxes} + \text{Real Estate Property Taxes} \\ &+ \text{State and Local Other Taxes}, \end{aligned} \tag{25}$$

and

$$\begin{aligned}\text{Capital Income} =&\text{Net Operating Surplus} + \text{Consumption of Private Fixed Capital}\\&+ (1-\alpha)\text{ Proprietors' Income,}\end{aligned} \tag{26}$$

where it is understood that the income flows, including net operating surplus, are measured net of their corresponding housing income flows. "State and Local Other Taxes" includes items like licensing fees.

3.9 The Return to Capital

As in Gomme et al. (2011), the after-tax return to capital can be computed from NIPA data by dividing after-tax private market capital income by the corresponding capital stock:

$$R_t = \left[\left(\frac{\text{(After-tax Capital Income)}/4}{\text{Market Capital Stock}}\right)^4 - 1\right] \times 100\% \tag{27}$$

where

$$\begin{aligned}\text{After-tax Capital Income} =&\text{Net Operating Surplus} - (1-\alpha)\text{ Proprietors Income}\\&-\tau_h\left[\text{Net Interest} - \alpha\text{ Proprietors Income} - \text{Rental Income}\right]\\&-\text{Taxes on Corporate Income} - \text{Business Property Taxes}\\&-\text{State and Local Other Taxes.}\end{aligned} \tag{28}$$

Since income is reported at an annual rate, the division of income by 4 in Eq. (27) expresses income at a quarterly rate.

3.10 The Solow Residual

Given data on hours of work, the capital stock and output, and armed with an estimate of α, capital's share of income, the Solow residual can be computed from the aggregate production function, Eq. (9).[7] Denote the Solow residual by Z_t. To obtain the properties of the productivity shock in Eq. (10), namely the autoregressive parameter, ρ, and the standard deviation of the innovation, σ, run the following regression:

$$\ln Z_t = \beta_0 + \beta_1 \ln Z_{t-1} + \beta_2 t + u_t. \tag{29}$$

[7] A time series for quarterly capital can be obtained from the reported annual capital stock data by using quarterly investment flows and imputing the depreciation rate; see Gomme and Rupert (2007) for details.

The time trend, t, is included to remove secular growth. The estimate of β_1 corresponds to ρ while the standard error of the residual corresponds to σ.

The results of estimating Eq. (29) over the period 1954Q1 through 2010Q4 are:

$$\ln Z_t = \underset{(0.1145)}{0.3252} + \underset{(0.0158)}{0.9555} \ln Z_{t-1} + \underset{(8.0482\times 10^5)}{3.4144 \times 10^5 t}, \tag{30}$$

standard errors in parentheses. The standard deviation of the residual is 0.00861.

3.11 Calibration Targets: Summary

Table 2 summarizes the implications of the calculations above for the calibration targets. In order to provide a single source of targets, Table 2 includes targets relevant for those interested in modeling the home sector, as well as a disaggregation of the depreciation rates into their constituent components (for the market sector, between structures and equipment & software; for the home sector, between structures (housing) and consumer durables).

A few comments are in order. First, the depreciation rate for market structures is substantially lower than that of equipment & software. The overall market depreciation rate is, clearly, a weighted average of the two components where the weights are given by the relative sizes of the two capital stocks. As shown in Gomme and Rupert (2007), while the market structures-output ratio is relatively constant in the post-World War II period, that of equipment & software has move up by roughly 10 percentage points.

Second, while the after-tax return to private market capital is somewhat higher than the 4% real return that the bulk of the macroeconomics literature calibrates to, the pre-tax return is much higher. The conventional justification for using a 4% return is that it represents a rough average of stock market returns, around 7% according to Mehra and Prescott (1985), and the return to a risk-free bond, 0.8% again according to Mehra and Prescott. Recall, though, that the return to private market capital computed above includes income flows that correspond to income from the stock market as well as bonds. What is potentially missing is intangible capital. Omitting intangible capital – which presumably earns a return measured in NIPA – biases up the measured return to private market capital. However, as reported in Gomme et al. (2011), the average return to the S&P 500 is somewhat *higher* than the return to private market capital, despite the fact that the prices of the stocks making up the S&P 500 should be pricing in the value of intangible capital.

Third, the properties of the technology shock are fairly similar to those of Prescott (1986): an autoregressive parameter of 0.95 and a standard deviation of the innovation of 0.00763.

Table 2: Calibration Targets

Target	Value
Risk aversion	$1-2$
Frisch labor supply elasticity	1.0
Time:	
Market, 16+	0.255
Market, $16-64$	0.315
Home, 16+	0.24
Home, $16-64$	0.251
Capital's share of income	0.2852
Depreciation rates:	
Market	0.0718
Structures	0.0289
Equipment & Software	0.1460
Home	0.0612
Housing	0.0159
Durables	0.2070
Investment-output ratio	0.1617
Capital-output ratio	1.6590
Labor tax rate	0.2417
Capital tax rate	0.4002
Return to capital:	
Pre-tax	9.4115
After-tax	4.9869
Technology shock:	
Autoregressive parameter	0.9555
Standard deviation of the residual	0.00861

Notes: Average time allocations are based on calculations by Gomme and Rupert (2007) from the American Time Use Survey. Capital's share of income, the depreciation rates, the ratios, tax rates and properties of the technology shock are based on data for the period 1954Q1 through 2010Q4; see the estimates in Eq. (30). The return to capital is averaged over 1954Q1 through 2009Q4.

3.12 Calibration In Action

A model period is set to one quarter. Recall from Table 1 that there are 9 parameters to be calibrated. Some of these parameters can be set directly from Table 2. These include: δ, the depreciation rate, based on the depreciation rate of market capital; α, capital's share of income; ρ and σ, the properties of the technology shock (see the estimates in Eq. (30)); and the tax rates on labor income, τ_n, and capital income, τ_k. For the purposes of this demonstration, the New Classical calibration will be followed, and the coefficient of relative risk aversion is set to 2. The remaining parameters are β, the discount factor, and ω which determines the importance of leisure in preferences. These parameters should be calibrated to "high quality" targets, and so are chosen so that the model's steady state delivers the observed fraction of time spent working, 25.5%, and an annual real return to capital of 4.9869%. A reasonable alternative to the return to capital would be the investment-output ratio. Given the discussion above concerning conceptual issues in obtaining the capital-output ratio, as well as the large – if infrequent – revisions to broad definitions of capital, either the return to capital or the investment-output ratio should be strictly preferred as calibration targets over the capital-output ratio.

The equations characterizing a solution to this model consist of:

$$U_1(c_t, 1-n_t)F_2(k_t, n_t; z_t) = U_2(c_t, 1-n_t) \tag{31}$$

$$U_1(c_{t+1}, 1-n_{t+1}) = \beta E_t \left\{U_1(c_{t+1}, 1-n_{t+1})\left[(1-\tau_k)F_1(k_{t+1}, n_{t+1}; z_{t+1}) + 1 - \delta\right]\right\} \tag{32}$$

$$c_t + k_{t+1} = F(k_t, n_t; z_t) + (1-\delta)k_t \tag{33}$$

Imposing the functional forms above, in steady state, these equations read:

$$\frac{1}{c}(1-\alpha)\left(\frac{k}{n}\right)^{\alpha} = \frac{\omega}{1-n} \tag{34}$$

$$1 = \beta\left[(1-\tau_k)\alpha\left(\frac{n}{k}\right)^{1-\alpha} + 1 - \delta\right] \tag{35}$$

$$c + \delta k = k^{\alpha} n^{1-\alpha} \tag{36}$$

These equations can be solved for c, n, k, ω and β, imposing the additional restrictions that

$$n = 0.255 \tag{37}$$

$$(1-\tau_k)\alpha\left(\frac{n}{k}\right)^{1-\alpha} + 1 - \delta = 1.049869^{\frac{1}{4}} \tag{38}$$

Doing so results in the following steady state values and parameter values:

$$k = 5.7659, \quad h = 0.255, \quad c = 0.5142, \quad \beta = 0.9879, \quad \omega = 2.5205. \tag{39}$$

In steady state, output is 0.6206. This calibration implies that the steady state consumption-output ratio is 0.8285, and that the annual capital-output ratio is 2.3228.

4 Simulation

While first-order perturbation methods are quite popular for solving dynamic general equilibrium models, here the model is solved by a policy function iteration method (also known as a projection method, or finite element method); see Coleman (1990) for details. This algorithm solves the Euler equations and constraints exactly as a set of grid points for the state variables, with linear interpolation between grid points.

In the business cycle literature, standard practice is to produce one table of second moments for the U.S. economy and a second such table for the model economy (reporting the average over many replications for the model economy). Here, instead, the model is simulated once using the measured innovations to the technology shock (Solow residual). Time series for the model are, then, compared to corresponding series from the U.S. data. In order to remove the secular trend from the data, apply the natural logarithm to the data, then apply the Hodrick and Prescott (1997) filter. The simulated data is similarly filtered.

Constructing quarterly time series is, at times, problematic. The principal problem is that some time series are only available annually, at least over part of the desired sample period of 1954Q1 through 2010Q4. Details concerning the various necessary manipulations can be gleaned from the Matlab/Octave file that constructs the data for this paper, or by referring to Gomme and Rupert (2007) and Gomme et al. (2011).

Figure 1a shows a remarkably good "fit" between the model's prediction for the path of output and that actually observed. This fit can also be seen in the scatter plot in Figure 2a where each dot represents a combination of actual and simulated output at some particular period of time. In particular, the scatter plot exhibits a positive association between actual and simulated, and a fairly tight fit as indicated by the fact that the points are reasonably tightly clustered along the 45 degree line.

The story is much the same for consumption, as seen in Figure 1b, although the model produces considerably less volatility in consumption since the mid-1980s than is in the data. Nonetheless, there is a high correlation between the actual and simulated consumption series, as seen in Figure 2b, although the correlation is not as tight as for output.

The fit between actual and simulated investment is weaker than seen for output and consumption. Figure 1c show that not only is simulated investment more volatile than actual investment, the correlation between the two series is much weaker. These observations are also borne out in the scatter plot, Figure 2c. The behavior of investment feeds into that of

Figure 1: Actual and Simulated Data

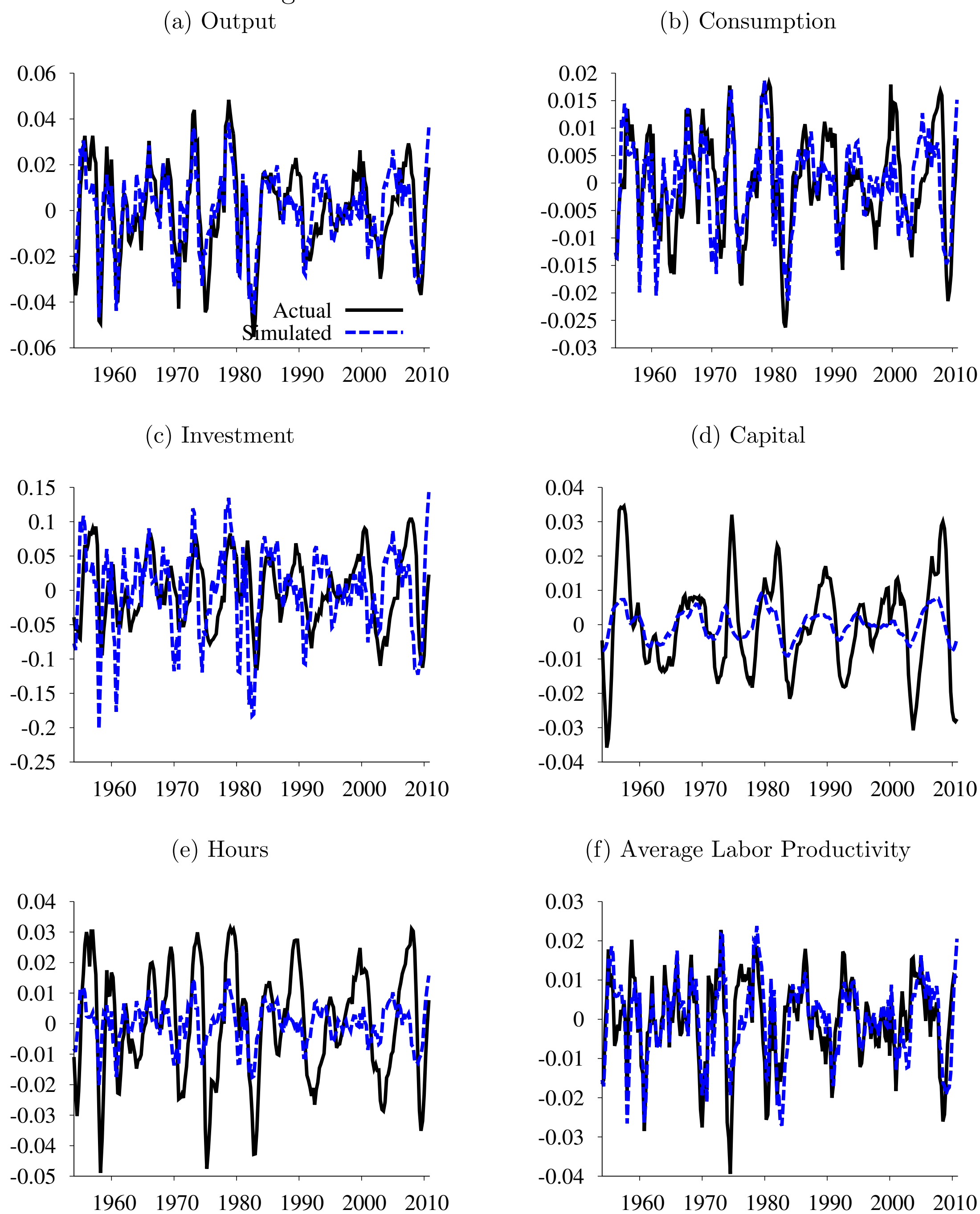
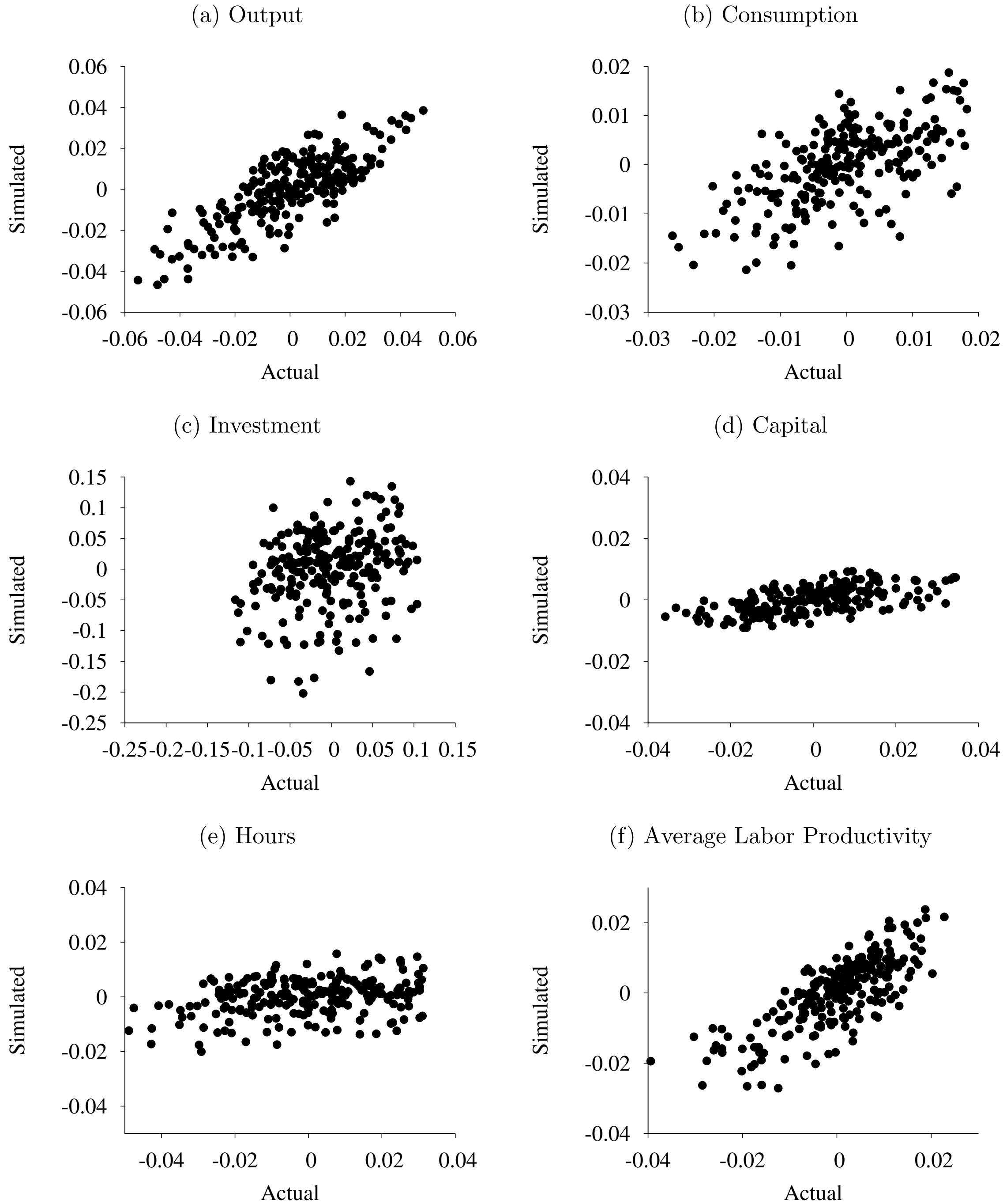


Figure 2: Actual and Simulated Data



capital; see Figure 1d. The scatter plot, Figure 2d, shows essentially no correlation between the actual and simulated capital stocks.

It is well known from the business cycle literature that models like the one in this paper deliver too little variability in hours worked. Figure 1e confirms this observation. Further, Figure 2e shows that there is no correlation between actual and simulated hours.

The model does reasonably well in mimicing the time series pattern of average labor productivity, output divided by hours; see Figure 1f. The correlation between actual and simulated productivity is fairly high; see Figure 2f. Given that the model does so poorly in predicting the time path of hours worked, its success with respect to productivity is, perhaps, surprising.

5 Concluding Remarks

This chapter presented, in detail, how to construct calibration targets. To make it easier for others to work with this data, a Matlab/Octave program file is available for download at `http://alcor.concordia.ca/~pgomme`. This program not only reports calibration targets, it also generates a basic set of data and computes business cycle moments

The chapter described how to actually calibrate the neoclassical growth model. Since there are more potential calibration targets than parameters, there is some discretion in choosing calibration targets. It was argued that well-measured, high quality calibration targets should be used whenever possible.

Finally, the model presented simulations of the model, comparing the model's predicted macroeconomic time series with those of the U.S.

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